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# Probability & Statistics


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# TEST OF HYPOTHESIS

There are many problems in which, rather then estimating the value of a parameter, we need to decide whether to accept or reject **statement** about the parameter. This statement is called **hypothesis** and the decision making procedure about the hypothesis is called **Hypothesis-Testing.**

For e.g.

a) The teaching method in both the institutes are effective

b) The average IQ of normal human being is 113

# Type of Hypothesis

a) **NULL HYPOTHESIS**

**b) ALTERNATIVE (or Research) HYPOTHESIS**

# Type of Hypothesis

## Null Hypothesis $H_0$

It is a statistical hypothesis that contains a statement of equality such as

$$\geq\ ,\ \leq\ or =$$

## Alternate Hypothesis $H_1$

It is a statement that must be **true** when $H_0$ is **false** and it contains a statement of inequality such as

$$<\ ,\ >\ or\ \neq$$

# General form of Hypothesis testing

Example (Type-1): $H_0 : \mu \leq k$ or $H_0 : \mu = k$

$H_1 : \mu > k$ $\qquad$ $H_1 : \mu > k$

(called Right -tailed test)

Example (Type-2): $H_0 : \mu \geq k$ or $H_0 : \mu = k$

$H_1 : \mu < k$ $\qquad$ $H_1 : \mu < k$

(called Left-tailed test)

Example (Type-3): $H_0 : \mu = k$

$H_1 : \mu \neq k$

(called Two-tailed test)

# Setting the Null and Alternative Hypothesis

**Example:**

State the null and alternative hypothesis for the following problem. A university publicizes that proportion of its students who graduate in 4 years is 82%.

**Ans:** The proportion 82% can be written as $p = 0.82$ contains the statement of equality and it becomes the null hypotheses. So

$$H_0: \; p = 0.82$$

$$H_1: \; p \neq 0.82$$

# Setting the Null and Alternate Hypothesis

**Example:**

A Manufacturer of new drug claims that his drug cures at Least 90% patients of certain disease. A prospective buyer has to take decision to buy or not to buy the new drug. State the two statements for the prospective buyer.

**Ans:** Let p be the proportion of the patients of this disease cured by the new drug out of those who are treated by the new drug. So

$$H_0: \; p \le 0.90$$

$$H_1: \; p > 0.90$$

# Testing of Hypothesis

- When we perform a hypothesis testing we make one of the two decisions:

  i) Reject the null hypothesis or
  ii) Accept the null hypothesis

- As our decision is based on the sample rather than the entire population there is always possibility we will make wrong decision.

# TYPES OF ERROR

In testing a statistical hypothesis, we will encounter two types of errors:

| | $H_0$ is True | $H_0$ is False or $H_1$ is True |
|---|---|---|
| Accept $H_0$ | **Correct decision** | Type II error |
| Reject $H_0$ | Type I error | **Correct decision** |

# TERMINOLOGY

- **Type I error**: The error is made when the null hypothesis rejected even it is true.

- **Type II error**: The error is made when the null hypothesis ($H_0$) accepted when it false (or $H_1$ is true).

- **Size of the test / Level of significance of the test / Alpha ($\alpha$):**

  $\alpha$ = Probability of committing Type-I error.

  = $P$(Rejecting $H_0$ when $H_0$ is true)

# TERMINOLOGY

- **Beta ($\beta$):**

  $\beta$ = Probability of committing Type-II error.

  $= P(\text{Accepting } H_0 \text{ when } H_1 \text{ is true})$

- **Power of the Test (1-$\beta$):**

  $1-\beta = P(\text{Rejecting } H_0 \text{ when } H_1 \text{ is true})$

- **Test Statistic:** The decision is made based on the value of some statistic and the corresponding statistic is called test statistic.

- **Critical or Rejection region:** The values of the test statistic on which the null hypothesis is rejected.

# Problem

A process for making steel pipe is under control if the diameter of the pipe has a mean of 3.0000 inches with a standard deviation of 0.0250 inch. To check whether the process is under control, a random sample of size $n$ = 30 is taken each day and the null hypothesis $\mu = 3.0000$ is rejected if $\bar{X}$ is less than 2.9960 or greater than 3.0040. Find

(a) The probability of a Type I error

(b) The probability of a Type II error when $\mu = 3.0050$ inches.

# Solution

**Solution:** $n$ = 30, $\mu$ = 3.0000, $\sigma$ = 0.0250

$$(a)\quad \alpha = P(\text{Type I error}) = P(\overline{X} < 2.9960 \text{ or } \overline{X} > 3.0040)$$

$$= P(\overline{X} < 2.9960) + P(\overline{X} > 3.0040)$$

$$= P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < \frac{2.9960-3.0000}{0.0250/\sqrt{30}}\right) + P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} > \frac{3.0040-3.0000}{0.0250/\sqrt{30}}\right)$$

$$= P\left(Z < -0.876\right) + 1 - P(Z < 0.876)$$

since $\dfrac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ will be random variable with approximately

standard normal distribution.

$$\alpha = F(-0.876) + 1 - F(0.876) = 2F(-0.876) = 0.381$$

# Solution

$$(b)\ \beta = P(\text{Type II error}) = P(2.9960 < \bar{X} < 3.0040)\ (\text{when } \mu = 3.0050)$$

$$= P\left(\frac{2.9960 - 3.0050}{0.0250/\sqrt{30}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{3.0040 - 3.0050}{0.0250/\sqrt{30}}\right)$$

$$= P(-1.97 < Z < -.219)$$

$$\beta = F(-.219) - F(-1.97)$$

$$= 0.4133 - 0.0244 = 0.3889$$

since $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ will be random variable with approximat ely standard normal distributi on.

# Problem

Suppose $X$ has Poisson distribution with parameter $\lambda$. To test $H_0 : \lambda = 0.1$ versus $H_1 : \lambda > 0.1$, the following procedure is used: A random sample of size 20 is taken from $X$ and $H_0$ is rejected if $\sum_{i=1}^{20} X_i \geq 4$.

Find $\alpha$ and $\beta$ (when $\lambda = 0.2$).

# Hypothesis Concerning On Mean when variance is known

## Test concerning on mean for large sample and known $\sigma$

Critical Regions for testing on population mean when $\sigma$ known)

| *Alternative hypothesis* | *Critical Region:* |
|---|---|
| $\mu < \mu_0$ | $C = \{ Z : Z < -z_\alpha \}$ |
| $\mu > \mu_0$ | $C = \{ Z : Z > z_\alpha \}$ |
| $\mu \neq \mu_0$ | $C = \{ Z : Z < -z_{\alpha/2}$ or $Z > z_{\alpha/2} \}$ |

where $Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$ is called the test statistic.

# Hypothesis Concerning On Mean when variance is known

## Note:

- In Previous Test we assume the sample size is large ( in practice, large means $n \geq 25$)

- If the sample size is small (say $n < 25$) and $\sigma$ is known then we can not use previous critical regions but if population is normal then same critical region will work.

# Hypothesis Concerning On Mean

**Example:** According to the norms established for mechanical aptitude test, persons who are 18 years old should average 73.2 with a standard deviation of 8.6. If 45 randomly selected persons of that age averaged 76.7, test the null hypothesis $\mu$ = 73.2 against the alternative hypothesis $\mu$ > 73.2 at the 0.01 level of significance.

**Solution:** Given $\mu_0 = 73.2$

1. ***Null hypothesis $H_0$*:** $\mu = 73.2$ ***Alternative hypothesis $H_1$*:** $\mu > 73.2$
2. ***Level of significance*:** $\alpha = 0.01$
3. ***Criterion*:** Using a normal approximation for the distribution of the sample mean we reject the null hypothesis when

$Z > z_\alpha = 2.33$ where

$$Z = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}.$$

# Hypothesis Concerning On Mean

4. ***Calculations*:** Given $\mu_0 = 73.2$, $\sigma = 8.6$, $n = 45$ and $\overline{X} = 76.7$

$$Z = \frac{76.7 - 73.2}{8.6/\sqrt{45}} = 2.73$$

5. ***Decision*:** Since $Z = 2.73 > 2.33$, the null hypothesis that $\mu = 73.2$ is rejected at the 0.01 level of significance.

# Hypothesis Concerning On Normal Mean when variance is unknown

**Test concerning on mean of Normal population for any Sample size and unknown $\sigma$**

Critical Regions for testing on normal mean when $\sigma$ unknown)

| *Alternative hypothesis* | *Critical Region:* |
|---|---|
| $\mu < \mu_0$ | $C = \{ T_{n-1} : T_{n-1} < -t_\alpha \}$ |
| $\mu > \mu_0$ | $C = \{ T_{n-1} : T_{n-1} > t_\alpha \}$ |
| $\mu \neq \mu_0$ | $C = \{ T_{n-1} : T_{n-1} < -t_{\alpha/2} \text{ or } T_{n-1} > t_{\alpha/2} \}$ |

where $T_{n-1} = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$ is called the test statistic.

# Hypothesis Concerning On Normal Mean when variance is unknown

## Note:

- **In Previous Test we assume the population has normal distribution.**

- **If the sample size is large (say $n \geq 25$), population is non normal and $\sigma$ is unknown then we can still use the previous critical regions. This means we can scrap the assumption normality when $n$ is large.**

# Hypotheses Concerning On Mean

**Example:** Test run with 6 models of an experimental engine showed that they operated for 24, 28, 21, 23, 32 and 22 minutes with a gallon of a certain kind of fuel. If the probability of Type I error is 0.01, is this evidence against a hypothesis that on the average this kind of engine will operate for at least 29 minutes per gallon with this kind of fuel? Assume **normality.**

**Solution:**

Given $n = 6$, $x_1 = 24$, $x_2 = 28$, $x_3 = 21$, $x_4 = 23$, $x_5 = 32$, $x_6 = 22$.

$$\Rightarrow \overline{X} = 25 \text{ and } S = 4.195$$

# Hypotheses Concerning On Mean

1. ***Null hypothesis $H_0$*: $\mu \geq 29$**
   ***Alternative hypothesis $H_1$*: $\mu < 29$**
2. ***Level of significance*: $\alpha = 0.01$**
3. ***Criterion*: Assuming the population is normal we can use the critical region as**

   $C = \{ T_{n-1} : T_{n-1} < -t_\alpha \}$ where

$$T_{n-1} = \frac{\bar{X} - \mu_0}{S / \sqrt{n}}.$$

# Hypothesis Concerning On Mean

**4.** ***Calculations*:** Given $\mu_0 = 29$, $S = 4.195$

$n = 6$, $\overline{X} = 25$, and $t_\alpha = t_{0.01} = -3.365$.

$$T_5 = \frac{25 - 29}{4.195/\sqrt{6}} = -2.336.$$

**5.** ***Decision*:** Since $T_5 = -2.336 > -3.365$, the null hypothesis that $\mu = 29$ is accepted at the 0.01 level of significance.

# Five steps of hypothesis testing

- We formulate a null hypothesis and an appropriate alternative hypothesis
- We specify the probability of a type I error; if possible, desired or necessary we may also specify the probabilities of Type II errors for particular alternatives.
- Based on the sampling distribution of an appropriate statistic, we construct a criterion for testing the null hypothesis against the given alternative.
- We calculate from the data the value of the statistic on which the decision is to be based.
- We decide whether to reject the null hypothesis or whether to fail to reject it.

**NOTE: Large sample (In practice, $n \geq 25$).**